ECE 583 Wireless Communications

This Week:

Random Variable
Gaussian Distribution
Rayleigh Distribution
Ricean Distribution

$\tilde{X} \rightarrow$ A random variable

$\tilde{X} \rightarrow$ can be discrete or continuous

If $\tilde{X}$ is a discrete R.V. $p(x) \rightarrow$ probability mass function of $\tilde{X}$

$$p(x) = \text{Prob}(\tilde{X} = x)$$

If $\tilde{X}$ is a continuous R.V. $f(x) \rightarrow$ probability density function of $\tilde{X}$

For discrete $\tilde{X}$ R.V.

$$E(\tilde{X}) = \sum_x x\,p(x) \quad \rightarrow \text{mean value of } \tilde{X}$$

$$\text{Var}(\tilde{X}) = E(\tilde{X}^2) - (E(\tilde{X}))^2$$

$$= \sum_x x^2 p(x) - \left(\sum x\,p(x)\right)^2$$

(2) If $\tilde{X}$ cont. RV.

$E(\tilde{X}) = \int x f(x)\, dx$ → mean value of $\tilde{X}$

$$\underbrace{Var(\tilde{X})}_{\text{variance of } \tilde{X}} = \underbrace{\int x^2 f(x)\, dx}_{E(\tilde{X}^2)} - (E(\tilde{X}))^2$$

If $\tilde{X}$ → is a Normal R.V. then

it has p.d.f $f(x) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right)$

and denoted by $\tilde{X} \sim N(m, \sigma^2)$

where m is the mean of $\tilde{X}$

i.e., $m = \int x f(x)\, dx$

and σ^2 is the variance of $\tilde{X}$

i.e., $\sigma^2 = E(\tilde{X}^2) - [E(\tilde{X})]^2$

σ → is called standard deviation of $\tilde{X}$

Note that $\sigma = \sqrt{Var(\tilde{X})}$

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Lets consider cont. R.V.s

$\tilde{X} \rightarrow$ cont. R.V.

function of $\tilde{X}$ is another R.V.

i.e, $\tilde{y} = g(\tilde{x})$

$\tilde{x}$ → R.V.

$\tilde{y}$ → R.V. $\quad g$ → function

Ex: $\tilde{y} = \tilde{x}^2 - 2\tilde{x} + 3$

If $\tilde{y} = g(\tilde{x})$ $\quad$ $\tilde{x}$ has p.d.f $f_{\tilde{x}}(x)$

$\tilde{y}$ has p.d.f $f_{\tilde{y}}(y)$

If $f_{\tilde{x}}(x)$ is known, then how to find $f_{\tilde{y}}(y)$

$\tilde{y} = g(\tilde{x}) \xrightarrow{\text{write}} y = g(x)$

Solve $y = g(x)$ and denote the roots as

$x_1, x_2, \cdots x_n$ $\quad$ i.e, $y = g(x_1) = g(x_2) = \cdots = g(x_n)$

then $f_{\tilde{y}}(y) = \dfrac{f_{\tilde{x}}(x_1)}{|g'(x_1)|} + \cdots + \dfrac{f_{\tilde{x}}(x_n)}{|g'(x_n)|} + \cdots$

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Ex:

$$\tilde{y} = e^{\tilde{x}}$$

Find $f(y)$ in terms of $f(x)$

Sln: $\tilde{y} = e^{\tilde{x}} \longrightarrow y = e^{x}$

$g'(x) = e^{x}$

x, y are in the range space of $\tilde{x}$ & $\tilde{y}$

If $y > 0$ then

$y = e^{x}$ has the single solution

$x = \ln y$

If $y < 0$ then no solution

$g'(x) = e^{x}$

$y_1 = e^{x_1}$

$x_1 = \ln y_1$

$g'(x_1) = e^{\ln y_1}$

$= y_1$

Thus $$f_{\tilde{y}}(y) = \frac{f_{\tilde{x}}(x_1)}{|g'(x_1)|}$$

$$= \frac{f_{\tilde{x}}(\ln y)}{y}$$

If $\tilde{x} \sim N(m, \sigma^2)$ then

$$f_{\tilde{y}}(y) = \frac{1}{y} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-(\ln y - m)^2/2\sigma^2}$$

which is called Lognormal distribution

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In summary:

If $\tilde{X} \sim N(m, \sigma^2)$ i.e., $f_{\tilde{X}}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}$

and $\tilde{y} = e^{\tilde{X}}$

then $\tilde{y}$ has p.d.f

$$f_{\tilde{y}}(y) = \frac{1}{y} \frac{1}{\sigma\sqrt{2\pi}} e^{-(\ln y - m)^2/2\sigma^2}$$

$\tilde{y}$ is Lognormally distributed.

i.e, $\tilde{y}$ has lognormal distribution

Two Random Variables:

$\tilde{X}, \tilde{y}$ are two RVs

$f(x,y) \rightarrow$ Joint p.d.f. of $\tilde{X}$ & $\tilde{y}$

$F(x,y) = \text{Prob}(\tilde{X} \leq x, \tilde{y} \leq y) \rightarrow$ Joint Cumulative density function of $\tilde{X}$ & $\tilde{y}$

$$f(x,y) = \frac{\partial F(x,y)}{\partial x \partial y}$$

Joint p.d.f.

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One function of Two RVs.

$\tilde{x}$ & $\tilde{y}$ are RVs

$$\tilde{z} = g(\tilde{x}, \tilde{y})$$

R.U.   R.U.   R.U.

function

Ex: $\tilde{z} = \tilde{x}\tilde{y} - 2\tilde{x} + \tilde{y}$

$$F_{\tilde{z}}(z) = \text{Prob}(\tilde{z} \leq z)$$
$$= \text{Prob}\{(\tilde{x}, \tilde{y}) \in D_{\tilde{z}}\}$$
$$= \iint_{D_{\tilde{z}}} f(x,y)\,dx\,dy$$

Ex: $\tilde{z} = \sqrt{\tilde{x}^2 + \tilde{y}^2}$

$$F_{\tilde{z}}(z) = \iint_{D_{\tilde{z}}} f(x,y)\,dx\,dy$$

The region $D_{\tilde{z}}$ is the circle $x^2 + y^2 \leq z^2$

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and $F_{\tilde{z}}(z)$ equals the probability masses in this circle

$z=\sqrt{x^2+y^2}$ Lets make use of coordinate change

If we move to polar coordinates from Cartesian coordinates

$$x = r\sin\theta$$
$$y = r\cos\theta$$

$$dx\,dy = r\,dr\,d\theta$$

Let

$$F_{\tilde{z}}(z) = \int_0^{2\pi}\int_0^{z} g(r)\, r\,dr\,d\theta$$

$$= 2\pi \int_0^{z} r\, g(r)\,dr \quad z>0$$

If $f(x,y) = \dfrac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2}$

i.e., $f(x,y) = f(x)\,f(y)$

$\tilde{x} \sim N(0,\sigma^2)$ $\tilde{y} \sim N(0,\sigma^2)$

then $F_{\tilde{z}}(z) = \dfrac{1}{\sigma^2}\displaystyle\int_0^{z} r e^{-r^2/2\sigma^2} dr = 1 - e^{-z^2/2\sigma^2}$

$z>0$

(8) $f_{\tilde{z}}(z) = \dfrac{\partial F_{\tilde{z}}(z)}{\partial z}$

$$f_{\tilde{z}}(z) = \frac{z}{\sigma^2} e^{-z^2/2\sigma^2} \quad z > 0$$

↓

Rayleigh distribution

Thus: if $\tilde{x} \sim N(0, \sigma^2)$ $\tilde{y} \sim N(0, \sigma^2)$

and $\tilde{z} = \sqrt{\tilde{x}^2 + \tilde{y}^2}$

then $f_{\tilde{z}}(z) = \dfrac{z}{\sigma^2} e^{-z^2/2\sigma^2}$

↓ Rayleigh distribution

<u>Note:</u> The expression

$$f_{\tilde{z}}(z) = \frac{z}{\sigma^2} e^{-z^2/2\sigma^2} \quad z > 0$$

can also be written as

$$f_{\tilde{z}}(z) = \frac{z}{\sigma^2} e^{-z^2/2\sigma^2} u(z)$$

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Ex: Let $f(x,y) = \frac{1}{2\pi\sigma^2} e^{-[(x-m_1)^2+y^2]/2\sigma^2}$

$\tilde{x}, \tilde{y}, \tilde{z}$ are RVs $\tilde{z} = \sqrt{\tilde{x}^2+\tilde{y}^2}$

$$f_{\tilde{z}}(z)\,dz = \iint_{\Delta D_z} f(x,y)\,dx\,dy$$

The region ΔD_z of the plane such that $z < \sqrt{x^2+y^2} < z+dz$ is a circular ring with inner radius z and thickness dz

with $x = z\cos\theta$ $y = z\sin\theta$

$$dx\,dy = z\,dz\,d\theta$$

It follows that

$$f_{\tilde{z}}(z)\,dz = \iint_{\Delta D_z} f(x,y)\,dx\,dy$$

$$= \frac{1}{2\pi\sigma^2}\int_0^{2\pi} e^{-[(z\cos\theta-m_1)^2+(z\sin\theta)^2]/2\sigma^2}\, z\,dz\,d\theta$$

Hence, $f_{\tilde{z}}(z) = \frac{z}{2\pi\sigma^2} e^{-(z^2+m_1^2)/2\sigma^2}\int_0^{2\pi} e^{z m_1\cos\theta/\sigma^2}\,d\theta$

This yields $f_{\tilde{z}}(z) = \frac{z}{\sigma^2} I_0\left(\frac{z m_1}{\sigma^2}\right) e^{-(z^2+m_1^2)/2\sigma^2}$

where $$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{x\cos\theta}\, d\theta$$

is the modified Bessel function

Two Functions of Two Random Variables:

$\tilde{x}$ & $\tilde{y}$ are two RVs

$\tilde{z} = g(\tilde{x}, \tilde{y}) \qquad \tilde{w} = h(\tilde{x}, \tilde{y})$

$\{\tilde{z} \le z, \tilde{w} \le w\} = \{(\tilde{x}, \tilde{y}) \in D_{zw}\}$

with z & w two given numbers, we denote by D_{zw} the region of the plane such that $g(x,y) \le z$ and $h(x,y) \le w$

Clearly $\{\tilde{z} \le z, \tilde{w} \le w\} = \{(\tilde{x}, \tilde{y}) \in D_{zw}\}$

$$F_{\tilde{z}\tilde{w}}(z,w) = \text{Prob}\{(\tilde{x}, \tilde{y}) \in D_{\tilde{z}\tilde{w}}\}$$

$$= \iint_{D_{\tilde{z}\tilde{w}}} f_{\tilde{x}\tilde{y}}(x,y)\, dx\, dy$$

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Joint density:

$$\tilde{z} = g(\tilde{x},\tilde{y}) \qquad \tilde{w} = h(\tilde{x},\tilde{y})$$

given $f_{\tilde{x},\tilde{y}}(x,y) \rightarrow$ joint p.d.f of $\tilde{x}$ & $\tilde{y}$

how to find $f_{\tilde{z},\tilde{w}}(z,w) \rightarrow$ joint pdf of $\tilde{z}$ & $\tilde{w}$

<u>Theorem:</u> To find $f_{\tilde{z},\tilde{w}}(z,w)$ we solve the system

$$g(x,y) = z \qquad h(x,y) = w$$

denoting by (x_n, y_n) its real roots

$$g(x_n,y_n) = z \qquad h(x_n,y_n) = w$$

we maintain that

$$f_{\tilde{z},\tilde{w}}(z,w) = \frac{f_{\tilde{x},\tilde{y}}(x_1,y_1)}{|J(x_1,y_1)|} + \dots + \frac{f_{\tilde{x},\tilde{y}}(x_n,y_n)}{|J(x_n,y_n)|}$$

where
$$J(x,y) = \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix}^{-1}$$

is the Jacobian matrix determinant

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Ex:

$\tilde{x}$ & $\tilde{y}$ are RVs

$$\tilde{r} = \sqrt{\tilde{x}^2 + \tilde{y}^2} \qquad \tilde{\Phi} = \arctan \frac{\tilde{y}}{\tilde{x}}$$

assume that $\tilde{r} \geq 0$ and $-\pi < \tilde{\Phi} \leq \pi$

find $f_{\tilde{r},\tilde{\Phi}}(r, \phi)$?

Sln:

$$\left.\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \phi &= \arctan \frac{y}{x} \end{aligned}\right\} \longrightarrow \text{Solve for } x \,\&\, y \text{ in terms of } r \,\&\, \phi$$

$$x = r \cos \phi$$
$$y = r \sin \phi \qquad r > 0$$

$$J(x,y) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} \end{vmatrix}^{-1}$$

$$= \begin{vmatrix} \cos \phi & -r \sin \phi \\ \sin \phi & r \cos \phi \end{vmatrix}^{-1} = \frac{1}{r}$$

$$f_{\tilde{r},\tilde{\Phi}}(r, \phi) = r \, f_{\tilde{x}\tilde{y}}(r \cos \phi, r \sin \phi) \qquad r > 0$$

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If $f_{\tilde{x},\tilde{y}}(x,y) = \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2}$

then $f_{\tilde{r},\tilde{\phi}}(r,\phi) = \frac{r}{2\pi\sigma^2} e^{-r^2/2\sigma^2}$ $\quad r>0 \quad |\phi|<\pi$

and 0 otherwise

$$f_{\tilde{r},\tilde{\phi}}(r,\phi) = \underbrace{\frac{r}{\sigma^2} e^{-r^2/2\sigma^2}}_{f(r)} \cdot \underbrace{\frac{1}{2\pi}}_{f(\phi)}$$

Ex:

$$\tilde{x}\cos\omega t + \tilde{y}\sin\omega t = \tilde{r}\cos(\omega t - \tilde{\phi})$$

$|\phi| < \pi$

$\tilde{x} \sim N(0,\sigma^2)$

$\tilde{y} \sim N(0,\sigma^2)$ $\quad \tilde{x}$ & $\tilde{y}$ are independent

$\tilde{\phi}$ is uniform in the interval $(-\pi,\pi)$

and $\tilde{r}$ has a Rayleigh distribution

Proof: $\quad x = r\cos\phi \quad y = r\sin\phi$

$$f(x,y) = f(x)\,f(y)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-y^2/2\sigma^2}$$

$$= \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2}$$

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Using Jacobian matrix we can show that

$$f_{\tilde{r},\tilde{\phi}}(r,\phi) = \frac{r}{2\pi\sigma^2} e^{-r^2/2\sigma^2}$$

$\tilde{r}, \tilde{\phi}$ are independent $f(r,\phi) = f(r)\, f(\phi)$

$$f(r) = \frac{r}{\sigma^2} e^{-r^2/2\sigma^2} \qquad f(\phi) = \frac{1}{2\pi}$$

$|\phi| < \pi$

$r > 0$

Notes: If the RVs $\tilde{r}$ & $\tilde{\phi}$ are Independent

$\tilde{r}$ has a Rayleigh distribution

and $\tilde{\phi}$ is uniform in the interval $(-\pi, \pi)$

then the RVs

$\tilde{x} = \tilde{r}\cos\tilde{\phi}$

$\tilde{y} = \tilde{r}\sin\tilde{\phi}$

are $N(0,\sigma^2)$ and independent

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Q-functions:

$\tilde{x} \sim N(0, \sigma^2)$ let $\sigma^2 = 1$

i.e., $\tilde{x} \sim N(0,1) \rightarrow f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$$Q(x) = \text{Prob}\{\tilde{x} > x\}$$

$$= \int_x^{\infty} f(t)\,dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2}\,dt$$

$$Q(0) = \frac{1}{2} \quad Q(\infty) = 0 \quad Q(-\infty) = 1$$

$$Q(-x) = 1 - Q(x)$$

Some bounds for $Q(x)$ function

$$Q(x) \leq \frac{1}{2} e^{-x^2/2}$$

$$Q(x) \leq \frac{1}{x\sqrt{2\pi}} e^{-x^2/2}$$

$$Q(x) > \frac{x}{(1+x^2)\sqrt{2\pi}} e^{-x^2/2}$$

For large x we have $Q(x) \approx \frac{1}{x\sqrt{2\pi}} e^{-x^2/2}$

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Complementary error function $erfc(x)$:

defined as

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2}\, dt$$

The complementary error function is related to the Q function as follows:

$$Q(x) = \frac{1}{2}\, erfc\left(\frac{x}{\sqrt{2}}\right)$$

$$erfc(x) = 2Q(\sqrt{2}\, x)$$

If $\tilde{x} \sim N(m, \sigma^2)$

then $\text{Prob}(\tilde{x} > \alpha) = Q\left(\frac{\alpha - m}{\sigma}\right)$

$$\text{Prob}(\tilde{x} < \alpha) = Q\left(\frac{m - \alpha}{\sigma}\right)$$

<u>More on Rayleigh distribution:</u>

If $\tilde{x}_1$ & $\tilde{x}_2$ are two iid $N(0, \sigma^2)$

then $\tilde{x} = \sqrt{\tilde{x}_1^2 + \tilde{x}_2^2}$

is a Rayleigh random variable

$$f_{\tilde{X}}(x) = \begin{cases} \dfrac{x}{\sigma^2} e^{-x^2/2\sigma^2} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

and its mean and variance are

$$E(\tilde{X}) = \sigma\sqrt{\frac{\pi}{2}}$$

$$Var(\tilde{X}) = \left(2 - \frac{\pi}{2}\right)\sigma^2$$

The CDF of a Rayleigh RV can be easily found by integrating the p.d.f.

The result is

$$F_{\tilde{X}}(x) = \begin{cases} 1 - e^{-x^2/2\sigma^2} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Gamma Function:

$\Gamma(x)$ defined as $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$

The Gamma function has simple poles at $x = 0, -1, -2, \ldots$ and satisfies the following properties

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$\Gamma(x+1) = x\Gamma(x)$

$\Gamma(1) = 1$

$\Gamma(1/2) = \sqrt{\pi}$

$$\Gamma\left(\frac{n}{2}+1\right) = \begin{cases} \left(\frac{n}{2}\right)! & n \text{ even \& positive} \\ \sqrt{\pi}\,\dfrac{n(n-2)(n-4)\ldots 3\times 1}{2^{\frac{n+1}{2}}} & n \text{ odd \& positive} \end{cases}$$

More on Bessel Functions

$I_\alpha(x) \rightarrow$ modified Bessel function of the first kind and order α

$$I_\alpha(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{\alpha+2k}}{k!\,\Gamma(\alpha+k+1)} \qquad x \geq 0$$

$$I_0(x) = \sum_{k=0}^{\infty} \left(\frac{x^k}{2^k k!}\right)^2$$

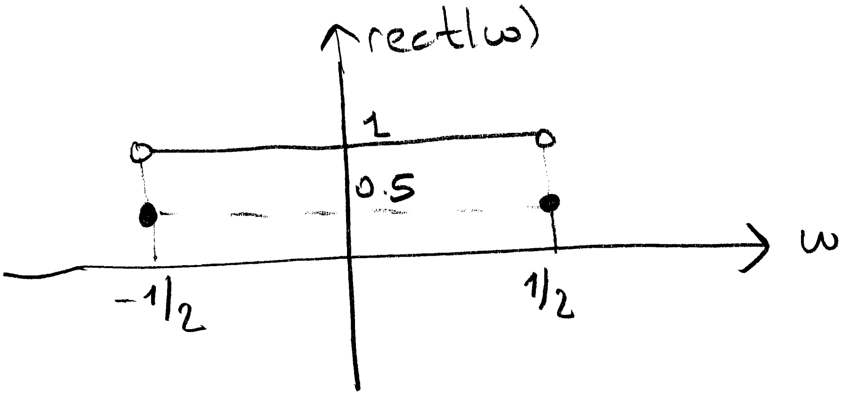
$$I_0(x) = \frac{1}{\pi}\int_0^{\pi} e^{\pm x\cos\phi}\,d\phi \qquad -\infty < x < \infty$$

for $x > 1$ $\quad I_0(x) \simeq \dfrac{e^x}{\sqrt{2\pi x}}$

$$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{x\cos\phi}\,d\phi$$

Rectangular Function:

$$\mathrm{rect}(\omega) = \Pi(\omega) = \begin{cases} 0 & \text{if } |\omega| > 1/2 \\ 1/2 & \text{if } |\omega| = 1/2 \\ 1 & \text{if } |\omega| < \frac{1}{2} \end{cases}$$



Fourier transform of $I_0(t)$, i.e., Bessel function of first kind of order 0

$$f(t) \xleftrightarrow{\text{Fourier transform}} F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt$$

Inverse Fourier transform $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t}\, d\omega$

$$I_0(t) \xrightarrow{\text{Fourier transform}} \sqrt{\frac{2}{\pi}}\, \frac{\mathrm{rect}(\omega/2)}{\sqrt{1-\omega^2}} \qquad \omega = 2\pi f$$

$$\sqrt{\frac{2}{\pi}}\, \frac{\mathrm{rect}(\pi f)}{\left(\sqrt{1 - 4\pi^2 f^2}\right)}$$

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Generalized Marcum Q function:

defined as

$$Q_m(a,b) = \int_b^{\infty} x\left(\frac{x}{a}\right)^{m-1} e^{-(x^2+a^2)/2} I_{m-1}(ax)\,dx$$

$$Q_1(a,b) = \int_b^{\infty} x\, e^{-\frac{a^2+x^2}{2}} I_0(ax)\,dx$$

Confluent Hypergeometric function:

$${}_1F_1(a,b;x) = \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b)x^k}{\Gamma(a)\Gamma(b+k)k!} \qquad b \neq 0,-1,-2\ldots$$

can be written in integral form as

$${}_1F_1(a,b;x) = \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)} \int_0^1 e^{xt} t^{a-1}(1-t)^{b-a-1}\,dt$$

The Chi-Square (χ^2) Random Variable:

$$\tilde{X}_i \sim N(0,\sigma^2) \qquad i=1,2,\ldots n$$

X_i are i.i.d RVs i.i.d → Independent and Identically distributed

We define $\tilde{X} = \sum_{i=1}^{n} \tilde{X}_i^2$

(20) then $\tilde{X}$ is a χ^2 random variable with n degrees of freedom.

The p.d.f. of this RV is given by

$$f(x) = \begin{cases} \frac{1}{2^{n/2}\,\Gamma\left(\frac{n}{2}\right)\sigma^n}\, x^{\frac{n}{2}-1}\, e^{-x^2/2\sigma^2} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$\Gamma(x)$ is the Gamma function defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

c.d.f of $\tilde{X}$ is

$$F_{\tilde{X}}(x) = \begin{cases} 1 - e^{-\frac{x^2}{2\sigma^2}} \sum_{k=0}^{m-1} \frac{1}{k!} \left(\frac{x}{2\sigma^2}\right)^k & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$E(\tilde{X}) = n\sigma^2$$

$$Var(\tilde{X}) = 2n\sigma^4$$

The Noncentral Chi-Square χ^2 R.V.

$\tilde{X}_i \sim N(m_i, \sigma^2) \qquad i = 1, \ldots n$

$$\tilde{X} = \sum_{i=1}^{n} \tilde{X}_i^2$$

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$$f(x) = \begin{cases} \dfrac{1}{2\sigma^2}\left(\dfrac{x}{s^2}\right)^{\frac{n-2}{4}} e^{-\frac{s^2+x}{2\sigma^2}} I_{\frac{n}{2}-1}\left(\dfrac{s}{\sigma^2}\sqrt{x}\right) & x>0 \\ 0 & \text{otherwise} \end{cases}$$

p.d.f of non-central χ^2

where s is defined as

$$s = \sqrt{\sum_{i=1}^{n} m_i^2}$$

$I_\alpha(x)$ is the modified Bessel function of the first kind and order α

For central χ^2 R.V.

if $n=2$ then

$$f(x) = \begin{cases} \dfrac{1}{2\sigma^2} e^{-x/2\sigma^2} & x>0 \\ 0 & \text{otherwise.} \end{cases}$$

In this case $E(\chi) = 2\sigma^2$

$$\text{i.e., } E(\chi) = \frac{1}{2\sigma^2}\int_0^\infty x e^{-x/2\sigma^2} dx$$

$$= 2\sigma^2$$

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Gamma Random variable.

$$f(x)=\begin{cases}\dfrac{\lambda(\lambda x)^{\alpha-1}e^{-\lambda x}}{\Gamma(\alpha)} & x\geq 0\\ 0 & \text{otherwise}\end{cases}$$

$\lambda, \alpha > 0$

A central χ^2 R.V. is a Gamma R.V.

with $\lambda=\frac{1}{2\sigma^2}$ and $\alpha=n/2$

The Ricean Random Variable:

$\tilde{X}_1 \sim N(m_1, \sigma^2)$ $\tilde{X}_2 \sim N(m_2, \sigma^2)$

$$\tilde{X}=\sqrt{\tilde{X}_1^2+\tilde{X}_2^2}$$

$\tilde{X}$ is a Ricean R.V. with p.d.f

$$f(x)=\begin{cases}\dfrac{x}{\sigma^2} I_0\left(\dfrac{sx}{\sigma^2}\right)e^{-\frac{x^2+s^2}{2\sigma^2}} & x>0\\ 0 & \text{otherwise}\end{cases}$$

where $s=\sqrt{m_1^2+m_2^2}$

for $s=0$ Ricean becomes Rayleigh

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$$E(\tilde{X}) = \sigma\sqrt{\frac{\pi}{2}}\ {}_1F_1\left(-\frac{1}{2}, 1, -\frac{s^2}{2\sigma^2}\right)$$

$$= \sigma\frac{\pi}{2}e^{-K/2}\left[(1+K)I_0\left(\frac{K}{2}\right)+KI_1\left(\frac{K}{2}\right)\right]$$

$$E(\tilde{X}^2) = 2\sigma^2 + s^2$$

$$K = \frac{s^2}{2\sigma^2}$$

If we define $A = s^2 + 2\sigma^2$ the Ricean p.d.f can be written as

$$f(x) = \begin{cases} \frac{2(K+1)}{A}x e^{-\frac{(K+1)}{A}\left(x^2+\frac{AK}{K+1}\right)} I_0\left(2x\sqrt{\frac{K(K+1)}{A}}\right) & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The Nakagami Random Variables:

p.d.f for this RV. is given as

$$f(x) = \begin{cases} \frac{2}{\Gamma(m)}\left(\frac{m}{\Omega}\right)^m x^{2m-1} e^{-mx^2/\Omega} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

where Ω is defined as $\Omega = E(\tilde{X}^2)$

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and the parameter m is defined as the ratio of moments, called the fading figure

$$m = \frac{\Omega^2}{E\left(\left(\tilde{X}^2 - \Omega\right)^2\right)} \qquad m \geq \frac{1}{2}$$

The mean and the variance for this R.V. are given by

$$E(\tilde{X}) = \frac{\Gamma(m+\frac{1}{2})}{\Gamma(m)} \left(\frac{\Omega}{m}\right)^{1/2}$$

$$Var(\tilde{X}) = \Omega\left(1 - \frac{1}{m}\left(\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m)}\right)^2\right)$$

Lognormal Random Variables

$\tilde{y} \sim N(m, \sigma^2) \qquad \tilde{X} = e^{\tilde{y}}$

$\tilde{X} \rightarrow$ is Lognormal R.V.

i.e., Logarithm of $\tilde{X}$ is Normal distributed

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$$E(\tilde{X}) = e^{m+\frac{\sigma^2}{2}}$$

$$Var(\tilde{X}) = e^{2m+\sigma^2}\left(e^{\sigma^2}-1\right)$$

Rayleigh & Rician Distributions (again)

$\tilde{X} \sim N(m_1, \sigma^2)$ $\quad\quad \tilde{z} = \tilde{X} + j\tilde{y}$

$\tilde{y} \sim N(m_2, \sigma^2)$

$$\tilde{R} = \sqrt{\tilde{X}^2 + \tilde{y}^2} \quad\quad \tilde{\Theta} = \tan^{-1}\left(\frac{\tilde{y}}{\tilde{X}}\right)$$

$\tilde{R} \rightarrow$ Envelope of $\tilde{z}$

$\tilde{\Theta} \rightarrow$ Phase of $\tilde{z}$

$$\left.\begin{array}{l} \tilde{R} = \sqrt{\tilde{X}^2 + \tilde{y}^2} \\ \tilde{\Theta} = \tan^{-1}\left(\frac{\tilde{y}}{\tilde{X}}\right) \end{array}\right\} \rightarrow \text{solve for } \tilde{X} \,\&\, \tilde{y}$$

$$\tilde{X} = \tilde{R}\cos\tilde{\Theta}$$

$$\tilde{y} = \tilde{R}\sin\tilde{\Theta}$$

$$J(x,y) = \begin{vmatrix} \frac{\partial x}{\partial R} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial R} & \frac{\partial y}{\partial \theta} \end{vmatrix}^{-1}$$

(26) $J(x,y) = \begin{vmatrix} \cos\theta & -R\sin\theta \\ \sin\theta & R\cos\theta \end{vmatrix}^{-1}$

$$= \left(R\cos^2\theta + R\sin^2\theta\right)^{-1}$$

$$= R^{-1}$$

$$= 1/R$$

$$f_{\tilde{R},\tilde{\theta}}(R,\theta) = \frac{f_{\tilde{x},\tilde{y}}(x_1,y_1)}{|J(x_1,y_1)|}$$

$f_{\tilde{x},\tilde{y}}(x,y) = f_{\tilde{x}}(x)\, f_{\tilde{y}}(y)$ $\quad \tilde{x}$ & $\tilde{y}$ are independent

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-m_1)^2/2\sigma^2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-m_2)^2/2\sigma^2}$$

$$= \frac{1}{2\pi\sigma^2} \exp\left[-\frac{(x-m_1)^2}{2\sigma^2} - \frac{(x-m_2)^2}{2\sigma^2}\right]$$

Thus

$$f(R,\theta) = R \cdot \frac{1}{2\pi\sigma^2} \exp\left[-\frac{(R\cos\theta - m_1)^2}{2\sigma^2} - \frac{(R\sin\theta - m_2)^2}{2\sigma^2} \right]$$

Consider the term

$$\frac{(R\cos\theta + m_1)^2}{2\sigma^2} + \frac{(R\sin\theta - m_2)^2}{2\sigma^2}$$

$$= \frac{R^2\cos^2\theta + m_1^2 - 2Rm_1\cos\theta + R^2\sin^2\theta + m_2^2 - 2Rm_2\sin\theta}{2\sigma^2}$$

$$= \frac{R^2 + (m_1^2 + m_2^2) - 2R(m_1\cos\theta + m_2\sin\theta)}{2\sigma^2}$$

Hence;

$$f(R,\theta) = \frac{R}{2\pi\sigma^2} \exp\left(-\frac{R^2+s^2}{2\sigma^2}\right) \exp\left(-R\,\frac{m_1\cos\theta + m_2\sin\theta}{\sigma^2}\right)$$

$s = \sqrt{m_1^2 + m_2^2}$ $\qquad \theta \in (-\pi, \pi]$ $\qquad R > 0$

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$$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{-x\cos\theta}\, d\theta$$

modified zeroth order Bessel function of the first kind

let $K = \frac{m_1^2+m_2^2}{2\sigma^2} = \frac{s^2}{2\sigma^2}$

$$f_{\tilde{R}}(R) = \int_{-\pi}^{\pi} f_{\tilde{R},\tilde{\Theta}}(R,\Theta)\, d\Theta$$

$$f_{\tilde{R}}(R) = \frac{R}{\sigma^2} \exp\left(-\frac{R^2+s^2}{2\sigma^2}\right) I_0\left(\frac{Rs}{\sigma^2}\right)$$

$R>0$

$$f_{\tilde{\Theta}}(\Theta) = \int_{-\infty}^{\infty} f_{\tilde{R},\tilde{\Theta}}(R,\Theta)\, dR$$

$$f_{\tilde{\Theta}}(\Theta) = \frac{1}{2\pi}\exp\left(-\frac{s^2}{2\sigma^2}\right) + \frac{\tilde{m}}{2\pi\sigma}\exp\left(\frac{\tilde{m}^2-s^2}{2\sigma^2}\right) Q\left(-\frac{\tilde{m}}{\sigma}\right)$$

$\Theta \in (-\pi, \pi]$

$$\tilde{m} = m_1\cos\Theta + m_2\sin\Theta = \sqrt{m_1^2+m_2^2}\cos\left(\Theta + \cos^{-1}\left(\frac{m_1}{m_2}\right)\right)$$

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$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp\left(-\frac{y^2}{2}\right) dy$$

When $K=0$ the marginal phase is

$$f_{\tilde{\Theta}}(\theta) = \begin{cases} \frac{1}{2\pi} & \theta \in [-\pi, \pi] \\ 0 & \text{otherwise} \end{cases}$$

i.e., $\tilde{\Theta}$ is uniformly distributed

When $K=0$, the Rician faded envelope reduces down to the Rayleigh faded envelope

$$f_{\tilde{R}}(R) = \frac{R}{\sigma^2} \exp\left(-\frac{R^2}{2\sigma^2}\right)$$

$$R > 0$$

Random Variable, Random Process, Auto Correlation Cross Correlation, White Sense Stationary Strict Sense Stationary,

- A random variable is a rule that assigns a number to every outcome of an experiment

This R.V. is associated with a sample space. To each event s in the sample space a number is assigned, and it is denoted by $\tilde{X}(s)$.

For stochastic processes (random processes) on the other hand, a time function $\tilde{X}(t,s)$ is assigned to every outcome in the sample space

Hence

$\tilde{X}(s) \longrightarrow$ R.V. (Random Variable)

$\tilde{X}(s,t) \longrightarrow$ Random Process (R.P)

For easy of the notation

$\tilde{X} \longrightarrow$ R.V.

$\tilde{X}(t) \longrightarrow$ R.P

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Ex:

$S = \{head, tail\}$ → Flip of a coin

$\tilde{X}(h) = 0.5$ $\quad p(h) = 0.5 \quad p(t) = 0.5$

$\tilde{X}(t) = 0.8$ $\quad \tilde{X} \rightarrow$ R.V.

$\tilde{y}(head, t) = 2t - 1$

$\tilde{y}(tail, t) = t^2 + 3$ $\quad \tilde{y} \rightarrow$ is a Random process

$\tilde{X}(s,t) \rightarrow$ for fixed t and variable s represent a random variable

For simplicity of notation we don't use s parameter and use $\tilde{X}(t)$ only for Random processes.

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$\tilde{X}(t)$ is a R.V. for a specific t

Hence $\tilde{X}(t)$ has a distribution.

Its cumulative distribution function is defined as

$$F(x,t) = \text{Prob}\{\tilde{X}(t) \leq x\}$$

Its derivative with respect to x is

$$f(x,t) = \frac{\partial F(x,t)}{\partial x}$$

which is the first order density of $\tilde{X}(t)$

i.e., prob. density function of $\tilde{X}(t)$

$\tilde{X}(t_1) \rightarrow$ R.V.

$\tilde{X}(t_2) \longrightarrow$ R.V.

Second order distribution of $\tilde{X}(t)$ is
(C.D.F)

given as

$$F(x_1, x_2; t_1, t_2) = \text{Prob}\{\tilde{X}(t_1) \leq x_1, \tilde{X}(t_2) \leq x_2\}$$

The corresponding density equals

$$f(x_1, x_2; t_1, t_2) = \frac{\partial^2 F(x_1, x_2; t_1, t_2)}{\partial x_1 \, \partial x_2}$$

Note that

$$f(x_1, t_1) = \int_{-\infty}^{\infty} f(x_1, x_2; t_1, t_2)\, dx_2$$

Second Order Properties of the Random Process $\tilde{X}(t)$

Mean: $m(t)$ of $\tilde{X}(t)$ is the expected value of the R.V. $\tilde{X}(t)$

$$m(t) = E\{\tilde{X}(t)\} = \int_{-\infty}^{\infty} x f(x, t)\, dx$$

Autocorrelation:

$$R(t_1, t_2) = E\{\tilde{X}(t_1)\tilde{X}(t_2)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f(x_1, x_2; t_1, t_2)\, dx_1 dx_2$$

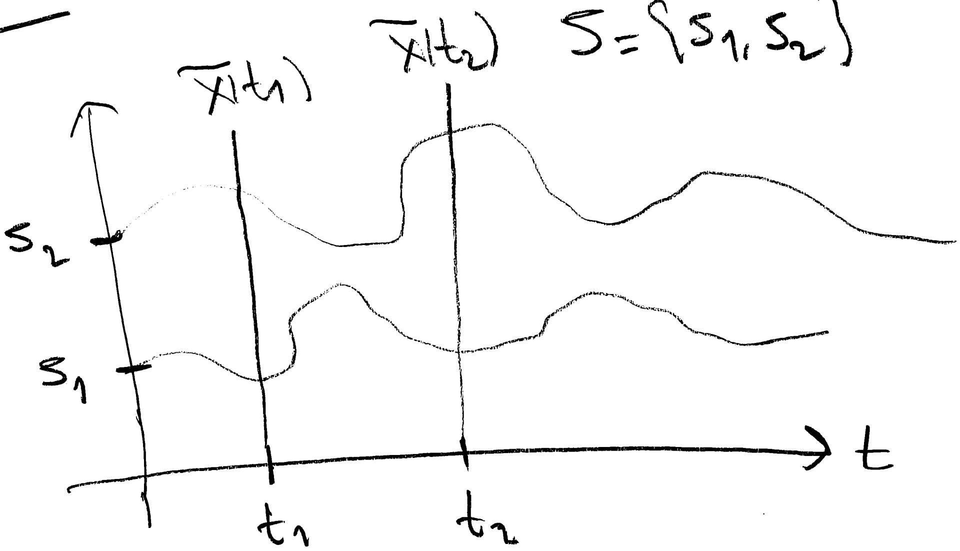
when $t_1 = t_2 = t$ $\quad R(t, t) = E\{\tilde{X}^2(t)\} \rightarrow$ average power of $\tilde{X}(t)$

Autocovariance:

The autocovariance $C(t_1, t_2)$ of $\tilde{X}(t)$ is the covariance of the RVs $\tilde{X}(t_1)$ & $\tilde{X}(t_2)$

$$c(t_1, t_2) = R(t_1, t_2) - m(t_1)m(t_2)$$

Example:



$\tilde{X}(t_1) \to$ R.V. $\quad \tilde{X}(t_1, s_1) = 0.2 \quad \tilde{X}(t_1, s_2) = 0.6$

$\tilde{X}(t_2) \to$ R.V. $\quad \tilde{X}(t_2, s_1) = 0.3 \quad \tilde{X}(t_2, s_2) = 0.9$

$$p(\tilde{X}(t_1) = 0.2) = 0.3 \qquad p(\tilde{X}(t_1) = 0.6) = 0.7$$

Let

$$p(\tilde{X}(t_2) = 0.3) = 0.4 \qquad p(\tilde{X}(t_2) = 0.9) = 0.6$$

$$m(t) = E\{\tilde{X}(t)\} = \int x \, f(x,t)\, dx$$

$$= \sum x \, p(x,t) \to \text{for discrete case}$$

$m(t_1) = 0.2 \times 0.3 + 0.6 \times 0.7$

$m(t_2) = 0.3 \times 0.4 + 0.9 \times 0.6$

(35) Ex:

$$\tilde{X}(t) = \tilde{R}\cos(\omega t + \tilde{\Phi})$$

$R(t_1, t_2)$?

$\tilde{R} \rightarrow$ R.V.

$\tilde{\Phi} \rightarrow$ uniform R.V. in the interval $(-\pi, \pi)$

$\tilde{R}$ & $\tilde{\Phi}$ are independent

Sln.:

$$R(t_1, t_2) = E\{\tilde{X}(t_1)\tilde{X}(t_2)\}$$

$$= E\{\tilde{R}\cos(\omega t_1 + \tilde{\Phi})\ \tilde{R}\cos(\omega t_2 + \tilde{\Phi})\}$$

$$= E\{\tilde{R}^2 \cos(\omega t_1 + \tilde{\Phi})\cos(\omega t_2 + \tilde{\Phi})\}$$

$$= E\left\{\tilde{R}^2 \frac{1}{2}\left[\cos(\omega t_1 + \omega t_2 + \tilde{\Phi} + \tilde{\Phi}) + \cos(\omega t_1 - \omega t_2)\right]\right\}$$

$$= \frac{1}{2}E(\tilde{R}^2)\left[E\{\cos(\omega t_1 + \omega t_2 + 2\tilde{\Phi})\} + E\{\underbrace{\cos(\omega t_1 - \omega t_2)}_{\text{constant}}\}\right]$$

$$= \frac{1}{2}E(\tilde{R}^2)\cos(\omega t_1 - \omega t_2) + \frac{1}{2}E(\tilde{R}^2)E\{\cos(\omega t_1 + \omega t_2 + 2\tilde{\Phi})\}$$

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$$E\{\cos(\omega t_1+\omega t_2+2\Phi)\} = \int_{-\pi}^{\pi} \cos(\omega t_1+\omega t_2+2\phi)\, \underbrace{f(\phi)}_{\substack{\phi \\ 1/2\pi \\ \text{uniform} \\ \text{pdf}}}\, d\phi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(\omega t_1+\omega t_2+2\phi)\, d\phi$$

$$= 0$$

Hence

$$R(t_1,t_2) = \frac{1}{2} E\{\hat{R}^2\} \cos(\omega(t_1-t_2))$$

if $\hat{R}$ is zero mean R.V. i.e, $\sigma^2 = E(\hat{R}^2)$

then $R(t_1,t_2) = \frac{1}{2}\sigma^2 \cos(\omega(t_1-t_2))$

Correlation Coefficient of Random Process

$\tilde{X}(t) \to$ R.P.

Autocovariance

$$C(t_1,t_2) = R(t_1,t_2) - m(t_1)\, m^*(t_2)$$

$$r(t_1,t_2) = \frac{C(t_1,t_2)}{\sqrt{C(t_1,t_1)\, C(t_2,t_2)}}$$ $\to$ correlation coefficient of RP $\tilde{X}(t)$

(37)

Complex process: (Complex R.P)

$\tilde{z}(t) = \tilde{x}(t) + j\tilde{y}(t)$

$\downarrow$ R.P. $\quad \downarrow$ R.P $\quad \downarrow$ R.P.

If $\tilde{x}(t)$ a complex process (R.P.)

i.e, $\tilde{x}(t) = \tilde{a}(t) + j\tilde{b}(t)$

then $R(t_1, t_2) = E\{\tilde{x}(t_1)\tilde{x}^{*}(t_2)\}$

$R(t_2, t_1) = R^{*}(t_1, t_2)$

Normal Processes:

A random process $\tilde{x}(t)$ is called normal if the RVs $\tilde{x}(t_1), \ldots \tilde{x}(t_n)$ are jointly normal for any n and $t_1, t_2, \ldots t_n$

The statistics of normal process are completely determined in terms of its mean $m(t)$ & auto covariance $c(t_1, t_2)$

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Notes

RVs $\tilde{x}$ & $\tilde{y}$ are jointly normal if their joint density is given by

$$f(x,y)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}}\exp\left\{-\frac{1}{2(1-r^2)}\left[\frac{(x-m_1)^2}{\sigma_1^2}-\frac{2r(x-m_1)(y-m_2)}{\sigma_1\sigma_2}+\frac{(y-m_2)}{\sigma_2^2}\right]\right\}$$

$|r|<1$

$$r=\frac{\operatorname{Cov}(\tilde{X}_1,\tilde{X}_2)}{\sigma_1\sigma_2}$$

$m_1=E\{\tilde{x}\}$
$m_2=E\{\tilde{y}\}$

Stationary Processes:

- A stochastic (Random) process $\tilde{x}(t)$ is called strict-sense stationary (SSS) if its statistical properties are invariant to a shift of the origin
  This means that $\tilde{x}(t)$ & $\tilde{x}(t+c)$ have the same statistics
- Two processes $\tilde{x}(t)$ & $\tilde{y}(t)$ are called jointly stationary if the joint statistics of $\tilde{x}(t)$ and $\tilde{y}(t)$ are the same as the joint statistics of $\tilde{x}(t+c)$ and $\tilde{y}(t+c)$ for any c

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– A complex process $\mathbf{z}(t) = \mathbf{x}(t) + j\mathbf{y}(t)$ is stationary if the processes $\mathbf{x}(t)$ & $\mathbf{y}(t)$ are jointly stationary

An SSS process must be such that

$$f(x_1, x_2, \ldots x_n; t_1, t_2, \ldots t_n) = f(x_1, x_2 \ldots x_n; t_1+c, \ldots t_n+c)$$

for any c

It follows that $f(x; t) = f(x; t+c)$

Hence the first-order density of $\mathbf{x}(t)$ is independent of t:

i.e., $f(x, t) = f(x)$

Similarly $f(x_1, x_2; t_1+c, t_2+c)$ is independent of c for any c. This leads to the conclusion that $f(x_1, x_2; t_1, t_2) = f(x_1, x_2; \tau)$ $\tau = t_1 - t_2$

Thus the joint density of the RVs $\mathbf{x}(t+\tau)$ and $\mathbf{x}(t)$ is independent of t and it equals $f(x_1, x_2; \tau)$

(40) Wide Sense Stationary

A stochastic process $\tilde{x}(t)$ is called wide-sense stationary (WSS) if its mean is constant

$$E\{\tilde{x}(t)\} = m$$

and its autocorrelation depends only on $t_1 - t_2$

$$R\{x(t_1)\, x^*(t_2)\} = R(t_1 - t_2)$$

If $\tau = t_1 - t_2$

then $$R\{x(t+\tau)\, x^*(t)\} = R(\tau)$$

$R(\tau)$ can be written in the symmetrical form

$$R(\tau) = E\left\{\tilde{x}\left(t+\frac{\tau}{2}\right) \tilde{x}^*\left(t-\frac{\tau}{2}\right)\right\}$$

$$R(0) = E\{|\tilde{x}(t)|^2\}$$ → average power of WSS process

- Two processes $\tilde{x}(t)$ & $\tilde{y}(t)$ are called Jointly WSS if each is WSS and their cross-correlation depends on $\tau = t_1 - t_2$

$$R_{xy}(t_1, t_2) = E\{\tilde{x}(t_1)\, \tilde{y}(t_2)\} = R_{xy}(t_1 - t_2)$$

 That is $R_{xy}(\tau) = E\{\tilde{x}(t+\tau)\,\tilde{y}(t)\}$

$$C_{xy}(\tau) = R_{xy}(\tau) - m_x m_y$$

The Power Spectrum:

In signal theory, spectra are associated with Fourier transforms.

<u>Defn:</u> The power spectrum (or spectral density) of a WSS process $\tilde{x}(t)$ real or complex, is the Fourier transform $S(\omega)$ of its autocorrelation

$$R(\tau) = E\{\tilde{x}(t+\tau)\,\tilde{x}^*(t)\}$$

i.e., $$S(\omega) = \int_{-\infty}^{\infty} R(\tau)\, e^{-j\omega\tau}\, d\tau$$

$S(\omega)$ is a real function of ω

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, e^{j\omega\tau}\, d\omega$$

Cross Power Spectrum (density)

$\tilde{x}(t)$ & $\tilde{y}(t)$ are two RPs (WSS)

$$R_{xy}(\tau) = E\{\tilde{x}(t+\tau)\,\tilde{y}(t)\}$$

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$R_{xy}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xy}(\omega)\, e^{j\omega\tau}\, d\omega$$

$S_{xy}(\omega)$ is complex quantity even when both $\tilde{x}(t)$ & $\tilde{y}(t)$ are real.

Table of Some Fourier Transforms

$$\delta(\tau) \xleftrightarrow{FT} 1$$

$$1 \xleftrightarrow{FT} 2\pi\,\delta(\omega)$$

$$e^{-\alpha|\tau|} \longleftrightarrow \frac{2\alpha}{\alpha^2+\omega^2}$$

$$\cos\beta\tau \longleftrightarrow \pi\left[\delta(\omega-\beta)+\delta(\omega+\beta)\right]$$

$$e^{-\alpha\tau^2} \longleftrightarrow \sqrt{\frac{\pi}{\alpha}}\, e^{-\omega^2/4\alpha}$$

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Digital Process (Random Process)

$\tilde{X}[n] \rightarrow$ Random Process

$n \in \mathbb{Z}$

Autocorrelation:

$$R[n_1, n_2] = E\{\tilde{X}[n_1]\,\tilde{X}^*[n_2]\}$$

Autocovariance:

$$C[n_1, n_2] = R[n_1, n_2] - m[n_1]\,m^*[n_2]$$

$\tilde{X}[n]$ is SSS

if $f_{\tilde{X}}[n] = f_{\tilde{X}}[n + n_0]$

$\tilde{X}$ is WSS

if $m[n] = \text{constant}$

$$R[n_1, n_2] = E\{\tilde{X}[n_1]\tilde{X}[n_2]\}$$

$$= R[n_1 - n_2]$$

if $n_1 - n_2 = m$

$$R[m] = E\{\tilde{X}[n+m]\,\tilde{X}^*[n]\}$$

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The Power Spectrum $\rightarrow$ P.S

$\tilde{X}[n] \rightarrow$ WSS

$$S(z) = \sum_{m=-\infty}^{\infty} R[m] z^{-m}$$

$$S(\omega) = \sum_{m=-\infty}^{\infty} R[m] e^{-jm\omega} \rightarrow \text{P.S.}$$

$$R[m] = \frac{1}{2\pi} \int_{-\pi}^{\pi} S(\omega) e^{jm\omega} d\omega$$

White Noise Process with Zero Mean

$\tilde{X}[n]$ is white noise with zero mean

if $m[n] = E\{\tilde{X}[n]\}$

$= 0$

and $R[n_1, n_2] = q[n_1] \delta[n_1 - n_2]$

where $\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$

and $q[n] = E\{\tilde{X}^2[n]\}$

If $\tilde{X}[n]$ is also stationary, then

$$R[m] = q\,\delta[m]$$

Thus a WSS white noise is a sequence of i.i.d. RVs with variance q.

A white noise process has constant power spectral density for all frequencies

this constant is usually denoted by $\frac{N_0}{2}$

Thermal Noises

Thermal noise is the noise generated in electronic devices by thermal agitation of electrons. Thermal noise can be closely modeled by a random process $N(t)$ having the following properties

1) $N(t)$ is a stationary process
2) $N(t)$ is a zero-mean process
3) $N(t)$ is a Gaussian process
4) $N(t)$ is a white noise process whose power spectral density is given by

$$S_N(f) = \frac{N_o}{2} = \frac{kT}{2}$$

where T is the ambient temperature in kelvins and k is Boltzmann's constant, equal to $1.38 \times 10^{-23} \frac{J}{K}$

Characteristic Functions

$\tilde{X} \longrightarrow$ R.V.

The characteristic function of an RV $\tilde{X}$ is by definition the integral

$$\Phi(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{j\omega x}\, dx$$

$$|\Phi(\omega)| \leq 1$$

Moment Generating Functions

If $s = j\omega$ in the characteristic function

then $$\Phi(s) = \int_{-\infty}^{\infty} f(x)\, e^{sx}\, dx$$

(47) Clearly $\Phi(\omega) = E\{e^{j\omega x}\}$

$$\Phi(s) = E\{e^{sx}\}$$